Galina Guzhvina (Westfaelische Wilhelms Universitaet, Muenster, Germany)
The Action of the Ricci Flow on Almost Flat Manifolds

`Abstract:` A compact Riemannian manifold M^n is called ε-flat if its curvature is bounded in terms of the diameter as follows:

$$|K| \leq \varepsilon \cdot d(M)^{-2},$$

where K denotes the sectional curvature and $d(M)$ the diameter of M: If one scales an ε-flat metric it remains ε-flat.

By almost flat we mean that the manifold carries ε-flat metrics for arbitrary $\varepsilon > 0$.

The (unnormalized) Ricci flow is the geometric evolution equation in which one starts with a smooth Riemannian manifold (M^n, g_0) and evolves its metric by the equation:

$$\frac{\partial}{\partial t} g = -2ric_g, \tag{1}$$

where ric_g denotes the Ricci tensor of the metric g.

When M^n is compact, one often considers the normalized Ricci flow

$$\frac{\partial}{\partial t} g = -2ric_g + \frac{2sc(g)}{n} g, \tag{2}$$

where $sc(g)$ is the average of the scalar curvature of M^n. Under the normalized flow the volume of the solution metric is constant in time, equations (1) and (2) differ only by a change of scale in space by a function of t and change of parametrization in time (see [?]).

The subject of my thesis was study how the Ricci flow acts on almost flat manifolds. We show that on a sufficiently flat Riemannian manifold (M, g_0) the Ricci flow exists for all $t \in [0, \infty)$, $\lim_{t\to\infty} |K|_{g(t)} \cdot d(M, g(t))^2 = 0$ as $g(t)$ evolves along (1), moreover, if $\pi_1(M, g_0)$ is abelian, $g(t)$ converges along the Ricci flow to a flat metric. More precisely, we establish the following result:

Main Theorem (Ricci Flow on Almost Flat Manifolds.)

In any dimension n there exists an $\varepsilon(n) > 0$ such that for any $\varepsilon \leq \varepsilon(n)$ an ε-flat Riemannian manifold (M^n, g) has the following properties:

(i) the solution $g(t)$ to the Ricci flow (1)

$$\frac{\partial g}{\partial t} = -2ric_g, \quad g(0) = g,$$

exists for all $t \in [0, \infty)$,

(ii) along the flow (1) one has

$$\lim_{t\to\infty} |K|_{g_t} \cdot d^2(M, g_t) = 0$$

(iii) $g(t)$ converges to a flat metric along the flow (1), if and only if the fundamental group of M is (almost) abelian (= abelian up to a subgroup of finite index).