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Signature of transitive Lie algebroids

`Abstract:` For each transitive Lie algebroid $(A, [\![\cdot,\cdot]\!], \#_A)$ with the anchor $\#_A : A \to TM$, over n-dimensional compact oriented manifold M and n-dimensional structure Lie algebras $\boldsymbol{g}_{|x}$ [then $\operatorname{rank} A = m + n$], the following conditions are equivalent (Kubarski-Mishchenko, 2004 [K-M2])

(1) $\mathbf{H}^{m+n}(A) \neq 0,$

(2) $\mathbf{H}^{m+n}(A) = \mathbb{R}$ and $\mathbf{H}(A)$ is a Poincaré algebra,

(3) there exists a global non-singular invariant [with respect to the adjoint representation of A] cross-section ε of the vector bundle $\bigwedge^n \boldsymbol{g}$ ($\boldsymbol{g} = \ker \#_A$),

(4) $\boldsymbol{g}$ is orientable and the module class $\theta_A = 0.$

The condition (3) implies that $\boldsymbol{g}_{|x}$ are unimodular and A is the so-called TUIO-Lie algebroid (Kubarski, 1996 [K1]). The scalar Poincaré product

$$\mathcal{P}_A^i : \mathbf{H}^i(A) \times \mathbf{H}^{m+n-i}(A) \to \mathbb{R},$$

$$(\alpha, \beta) \longmapsto \int_A^{\#} \alpha \wedge \beta$$

is defined via the fibre integral $\fint_A : \Omega^\bullet(A) \to \Omega_{dR}^{\bullet - n}(M)$ by the formula

$$\int_A^{\#} \alpha \wedge \beta = \int_M \left(\fint_A^{\#} \alpha \wedge \beta \right).$$

The condition (3) implies that the operator $\fint_A$ commutes with the differentials d_A and d_M giving a homomorphism in cohomology $\fint_A^{\#} : \mathbf{H}^\bullet(A) \to \mathbf{H}_{dR}^{\bullet - n}(M).$ In particular we have $\fint_A^{\#} : \mathbf{H}^{m+n}(A) \to \mathbf{H}_{dR}^m(M) = \mathbb{R}$. The scalar product $\mathcal{P}_A^i$ is nondegenerated and if $m + n = 4k$ then

$$\mathcal{P}_A^{2k} : \mathbf{H}^{2k}(A) \times \mathbf{H}^{2k}(A) \to \mathbb{R}$$

is nondegenerated and symmetric. Therefore its signature is defined and is called the signature of A, and is denoted by $\operatorname{Sig}(A)$.

The problem is:

- to calculate the signature $\operatorname{Sig}(A)$ and give some conditions to the equality $\operatorname{Sig}(A) = 0$.

My talk concerns this problem.

(I) Firstly, I give a general mechanism of the calculation of the signature via spectral sequences (Kubarski-Mishchenko [K-M1]). Namely, under some simple regularity assumptions on a DG-algebra C^r we have: if $E_2^{j,i}$ leave in a finite rectangular and is a Poincaré algebra then

$$\operatorname{Sig}(E_2) = \operatorname{Sig}(\mathbf{H}(C)).$$

We use this mechanism to

(a) the spectral sequence for the Čech-de Rham complex of the Lie algebroid A proving that the trivial monodromy implies $E_2^{j,i} = \mathbf{H}^j(M) \otimes \mathbf{H}^i(\mathfrak{g})$ [$\mathfrak{g} = \boldsymbol{g}_{|x}$] and so $\operatorname{Sig}(A) = 0$ (such a situation take place for example if the structure Lie algebra $\mathfrak{g}$ is simple algebra of the type $B_l, C_l, E_7, E_8, F_4, G_2$).

(b) the Hochschild-Serre spectral sequence. For this sequence $E_2^{j,i} = \mathbf{H}^j\left(M, \mathbf{H}^i(\boldsymbol{g})\right)$ with respect some flat connection in the vector bundle $\mathbf{H}^i(\boldsymbol{g})$ which enables us to use the Hirzebruch formula.

(II) Secondly, introducting a Riemannian structure in A we can define the $*$-Hodge operator, the codifferential and the signature operator and prove the Signature Theorem.

References

[K-M1] J.Kubarski, A. Mishchenko, *Lie Algebroids: Spectral Sequences and Signature,* Matem. Sbornik 194, No 7, 2003, 127-154.

[K-M2] J.Kubarski, A. Mishchenko, *Nondegenerate cohomology pairing for transitive Lie algebroids, characterization,* Central European Journal of Mathematics Vol. 2(5), p. 1-45, 2004, 663-707.

[K1] J.Kubarski, *Fibre integral in regular Lie algebroids,* New Developments in Differential Geometry, Budapest 1996, KLUWER ACADEMIC PUBLISHERS, Dordrecht, 1999, 173-202. Proceedings of the Conference on Differential Geometry, Budapest, Hungary, 27-30 July 1996.