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On a Blaschke problem in web theory (some example of webs formed by pencils of spheres)

Abstract: In every point of 3-dimensional space E^3, 4 pencils of spheres induce a configuration consisting of 4 sphere and 6 circles. So, 4 pencils of spheres form in E^3 a unique spherical 4-web W and six 3-webs each of them is formed by 2 pencils of spheres and 1 congruence of circles.

The sphere $a(x^2 + y^2 + z^2) + bx + cy + dz + e = 0$ we consider as a point (a, b, c, d, e) in projective space P^4 (Darboux representation).

Then, the pencil of spheres is straight line in P^4.

Example 1. Let S_1, S_2, S_3 be mutually ortogonal spheres, A and B be the common points of S_1,S_2, S_3. The pencils AS_1, S_1S_2, S_2S_3, AB form spherical web W_1 whose equation can be written as $1 + y^2 + y^2z^2 + ux^2 = 0$, or after an isotopic transformation as

$$xy + uv = 1. \tag{1}$$

This web is hexagonal but not regular (parallelizable).

Example 2. Spherical 4-web W_2 formed by pencils AS_1, S_1S_2, S_2S_3, S_3B. Its equation is $xyzu - y^2 - z^2 = 1$. The web W_2 is not hexagonal.

Example 3. Spherical 4-web W_0 with the following property: every sphere of W_0 is orthogonal to a sphere S_0. In other words, the corresponding straight lines l_i, $i = 1, 2, 3, 4$, in P^4 are situated in a 3-plane π. The web W_0 is hexagonal, but not regular. It is regular if the lines l_i form a closed cycle.

Example 4. 3-web W_4 formed by 2 elliptic pencils of spheres S_1S_2 and S_2S_3 (see Example 1) and congruence of circles generated by hiperbolic pencil of spheres AB and parabolic pencil AS_1. The equation of the web W_4 is also equation (1) where x and y are the parameters of two first families of spheres (S_1S_2 and S_2S_3), and (u, v) (mutually!) are the parameters of the congruence of circles. After isotopic transformation $\ln x \to x$, $\ln y \to y$, $\ln(1 - uv) \to -u - v$ we transform the equation (1) to the equation $x + y + u + v = 0$. So, 3-web W_4 is regular.