Alexey Rybnikov (Moscow State M.V. Lomonosov University, Moscow, Russia)

Theory of connections and Backlund maps for second-order partial differential equations

`Abstract:` In the present work the theory of Backlund transformations is treated as a special chapter of the theory of connections. Following F. Pirani and D. Robinson we consider the notion of Backlund transformation as a more general notion of *Backlund map*. We present here new interpretation of Backlund map. In this lecture Backlund maps are associated with the connections defining the representations of zero curvature for a given differential equation.

Remind first of all that Backlund transformations arose for the first time in 1879 as transformations of the surfaces with constant negative curvature in 3-dimensional Euclidean space $\mathbb{R}^3$ (wellknown BianchiLie transformation). In 1880 A. Backlund noticed that one can consider BianchiLie transformation as a particular case of the more general transformations defined by the system of four relations of the form

$$F_\alpha(x^1, x^2, z, z_1, z_2, x^{1'}, x^{2'}, y, y_{1'}, y_{2'}) = 0 \quad (\alpha = 1, 2, 3, 4),$$

where $z_i = \dfrac{\partial z}{\partial x^i}$ $(i = 1, 2)$, $y_{i'} = \dfrac{\partial y}{\partial x^{i'}}$ $(i' = 1', 2')$. Remark that A. Backlund treated these transformations as the mappings between the pairs of surfaces in $\mathbb{R}^3$. It was G. Darboux who noticed that Backlund transformation may be regarded as a mapping between integral manifolds of a pair of partial differential equations (PDE) or of a single differential equation. In 1977 F. Pirani and D. Robinson [1] presented an another interpretation of Backlund transformations. They treated the Backlund transformations as a particular case of more general notion of *Backlund maps*. In this lecture our interpretation of Backlund maps is distinguished from one of F. Pirani and D. Robinson.

We investigate here the Backlund maps for secondorder PDE

$$\Phi(x^1, x^2, z, z_1, z_2, z_{11}, z_{12}, z_{22}) = 0, \tag{1}$$

where $z_{kl} = \dfrac{\partial^2 z}{\partial x^k \partial x^l}$. While considering the equation (1), we assumed that x^1, x^2, z are adapted local coordinates on $(2+1)$-dimensional general type bundle H over 2-dimensional base (the variables x^1, x^2 are local coordinates on the base). Along with the manifold H we consider 1-jet manifold J^1H and 2-jet manifold J^2H. Denote the local coordinates of J^1J by x^i, z, p_j and the local coordinates of J^2H by x^i, z, p_j, p_{kl} $(p_{kl} = p_{lk})$. For any section $\sigma \subset H$ defined by the equation $z = z(x^1, x^2)$ one can consider the lifted sections $\sigma^1 \subset J^1H$ and $\sigma^2 \subset J^2H$ defined by the equations $z = z(x^1, x^2)$, $p_j = z_j$ and $z = z(x^1, x^2)$, $p_j = z_j$, $p_{kl} = z_{kl}$ accordingly. A somewhat more general form of the equation (1) is

$$\Phi(x^1, x^2, z, p_1, p_2, p_{11}, p_{12}, p_{22}) = 0. \qquad \text{(1bis)}$$

On the lifted section $\sigma^2 \subset J^2H$ of any section $\sigma \subset H$ this equation has the form (1). We say that a section $\sigma \subset H$ is a solution of the PDE (1) if the equation (1 bis) is identically satisfied on the lifted section $\sigma^2 \subset J^2H$.

One can consider the principal bundles over the base J^1H and over the base J^2H as well as associated bundles. Following [2] we denote $P(B, G)$ the principal bundle over the base B with the structural Lie group G. An associated bundle with the model fiber $\mathfrak{J}$ ($\mathfrak{J}$ is the representation space of the group G) we denote by $\mathfrak{J}(P(B, G))$. The bundle ${}^1RH = P(J^1H, SL(2))$ (it is a factor manifold of the 1-frame manifold) and the bundle ${}^2RH = P(J^2H, SL(2))$ (it is a factor manifold of the 2-frame manifold) are the most important for us. The bundles $P(J^kH, G)$ $(k = 1, 2)$, where G is a subgroup of $SL(2)$, are interested for us too.

One can consider the *special connections* [3] in the principal bundles $P(J^kH, G)$ $(k = 1, 2)$ $(G \subseteq SL(2))$. They generate corresponding connections in associated bundles and, in particular, in the associated bundles with one-dimensional fiber $\mathfrak{J}$. We say that a connection in $\mathfrak{J}(P(J^kH, G))$ $(\dim\mathfrak{J} = 1)$, where $(G \subseteq SL(2))$, is *Backlund connection of class k* corresponding to a given PDE (1) if it is generated by special connection defining the representation of zero curvature for the equation (1). Moreover note that in this case Pfaff equation

$$ {}_\sigma\theta = 0 \tag{2} $$

is completely integrable if and only if the section $\sigma \subset H$ is a solution of the equation (1). Here ${}_\sigma\theta$ is the connection form for the Backlund connection of the equation (1) considered on a section $\sigma \subset H$. The Pfaff equation (2) defines a mapping that takes each solution $\sigma \subset H$ of the equation (1) to the section ${}_\sigma\sum \subset \mathfrak{J}(P(J^kH, G))$ $(\dim\mathfrak{J} = 1)$ which is a solution of the Pfaff equation (2). We say that this mapping is *the Backlund map of class k* corresponding to the PDE (1). The Pfaff equation (2) we call *the Pfaff equation defining the Backlund map*.

The assignment of the Backlund map of class k of general type is equivalent to the assignment of the Backlund connection in $\mathfrak{J}({}^kRH)$ $(\dim\mathfrak{J} = 1)$. Moreover here $G = SL(2)$. In this case the Pfaff equation (2) (under special choice of the structural forms) is equivalent to the system of PDE

$$ y_i = -{}_\sigma\gamma^2_{1i} + y \cdot_\sigma \gamma_i + y^2 \cdot_\sigma \gamma^1_{2i} \quad (i = 1, 2), \tag{3} $$

where ${}_\sigma\gamma_i$, ${}_\sigma\gamma^2_{1i}$, ${}_\sigma\gamma^1_{2i}$ are the coefficients of the special connection in ${}^kRH$ (considered on a section $\sigma \subset H$) defining the representation of zero curvature for a given PDE (1). The coefficients γ_i, γ^2_{1i}, γ^1_{2i} depend on x^i, z, p_j if $k = 1$ and on x^i, z, p_j, p_{kl} if $k = 2$. The system (3) we call *the Backlund system*.

One can prove that the coefficients of special connection in ${}^2RH$ defining the representation of zero curvature for the secondorder PDE have the following form

$$ \gamma_i = \varphi \cdot p_{1i} + \psi \cdot p_{2i} + \chi_i, \; \gamma^2_{1i} = \varphi^2_1 \cdot p_{1i} + \psi^2_1 \cdot p_{2i} + \chi^2_{1i}, \; \gamma^1_{2i} = \varphi^1_2 \cdot p_{1i} + \psi^1_2 \cdot p_{2i} + \chi^1_{2i}, $$

where $\varphi, \varphi_1^2, \varphi_2^1, \psi, \psi_1^2, \psi_2^1, \chi_i, \chi_{1i}^2, \chi_{2i}^1$ are the functions of the variables x^1, z, p_j. Thus the Backlund system defining the Backlund map of class 2 is essentially more specific than one defining the Backlund map of class 1.

Along with the general Backlund connections one can consider the Backlund connections defined in the associated bundles $\mathfrak{J}(P(J^k H, G))$ $(\dim\mathfrak{J} = 1)$, where G is a subgroup of $SL(2)$. In this case the Backlund maps of more special types arise. Such Backlund maps are, in particular, the Backlund maps for evolution equations [3,4]. One can consider also the Backlund connections defined in $\mathfrak{J}(P(J^k H, G_1))$ $(\dim\mathfrak{J} = 1)$, where G_1 is one-dimensional subgroup of $SL(2)$. We call these connections *the ColeHopf connections*. The corresponding Backlund maps we call *ColeHopf maps* (or *CHmaps*). The very first example of such maps is wellknown transformation that takes the solutions of the Burgers equation $z_1 + z \cdot z_2 - z_{22} = 0$ to the solutions of the heat equation $y_1 = y_{22}$. This transformation has appeared at first in the works of J.D. Cole and E. Hopf (see [5] or original papers of J.D. Cole [6] and E. Hopf [7]). Remark that the assignment of the ColeHopf map is equivalent to the assignment of the potential of a given PDE.

In the present work we prove that if the secondorder PDE admits the Backlund maps of class 1 then it is quasilinear equation. If the secondorder equation admits the Backlund map of class 2 then it is either quasilinear equation or MongeAmpere type equation

$$z_{11} \cdot z_{22} - (z_{12})^2 + P z_{11} + Q z_{12} + R z12 + S = 0,$$

where P, Q, R, S depend on x^i, z, z_j.

It is obtained a number of conditions for existence of general Backlund maps for secondorder PDE. It is found also some conditions of existence of CHmaps (and consequently potentials). In particular, it is derived the necessary and sufficient conditions for existence of CHmaps of class 1 for the equations of the form $z_{11} + z_{22} + f(x^1, X62, z, z_1, z_2) = 0$ and for the equations of the form $z_{22} + f(x^1, x^2, z, z_1, z_2)$. Besides it is proved the existence of CHmaps of class 2 for MongeAmpere type equations of the form $z_{11} \cdot z_{22} - (z_{12})^2 + g(x) \cdot f(z_1, z_2) = 0$.

References

[1] *Pirani F.A.E., Robinson D.C.*, Sur la definition des transformations de Backlund. // C.R. Acad. Sc. Paris, Serie A. 1977. **T. 285.** P. 581-583.

[2] *Evtushik L.E., Lumiste U.G., Ostianu N.M., Shirokov A.P.*, The differential-geometric structures on the manifolds. // Itogi nauki. Problems in Geometry (VINITI, Moscow). 1979. **Vol. 9.** P. 5-246 (in Russian).

[3] *Rybnikov A.K.*, On special connections defining the representations of zero curvature for secondorder evolution equations. // Izv. Vuz. Math. **1979.** No. 9. P. 32-41 (in Russian).

[4] *Rybnikov A.K.*, The theory of connections and the problem of the existence of Backlund transformations for secondorder evolution equations. // Doklady Mathematics. **Vol. 71.** No. 1. 2005. P. 71-74.

[5] *Ablowitz M.J., Segur H.*, Solitons and the inverse scattering transform. Philadelphia: SIAM, 1981.

[6] *Cole J.D.*, On a quasilinear parabolic equation occurring in aerodynamics. // Quart. App. Math. **V. 9.** 1951. P. 225-236.

[7] *Hopf E.*, The partial differential equation . // Comm. Pure Appl. Math. 1950. P. 201-230.