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Soliton systems and Zermelo condition in higher-order space

Abstract: In this study, we discuss a relation between nonlinear physical fields called soliton systems and a differential geometric space called Kawaguchi space or higher-order space.

In Kawaguchi space, an arc length is given by $ds = F(x^i, x^{(1)i}, \cdots, x^{(M)i})dt$, where $i = 1, 2, \cdots n$. (x^i) is a coordinate system in the n-dimensional configuration space, $x^{(\alpha)i} = d^\alpha x^i/dt^\alpha$ and t is a parameter such as a time. The function F is the Lagrangian of order M. The arc length remains unaltered by a change of the parameter t. Therefore, in Kawaguchi space, the Lagrangian should satisfy the following condition (Zermelo condition):

$$\Delta_k F \equiv \sum_{\alpha=k}^{M} \binom{\alpha}{k} x^{(\alpha-k+1)i} \frac{\partial F}{\partial x^{(\alpha)i}} = \delta_k^1 F, \ k = 1, \cdots M. \tag{1}$$

Geometrically, some field equations in physics can be expressed in terms of Kawaguchi space. For example, Klein-Gordon equation and Dirac equation in quantum field theory have been derived from Zermelo condition. Based on this approach, we show a relation between Kawaguchi space and nonlinear physical systems called soliton systems. In order to obtain a nonlinear expression, we introduce Zermelo operator Δ_1' in 2-dimensional case as follows

$$\Delta_1' = \psi \frac{\partial}{\partial \psi} + \psi_i \frac{\partial}{\partial \psi_i} + \psi_{ii} \frac{\partial}{\partial \psi_{ii}} + \cdots, \tag{2}$$

where $\psi = \psi(x, t)$, $\psi_i = \partial\psi/\partial x^i$ and $\psi_{ii} = \partial^2\psi/\partial x^i \partial x^i$. Then, for a certain Lagrangian F, a soliton equation can be obtained from Zermelo condition $\Delta_1' F = F$. For example, KdV equation $\psi_t + 6\psi\psi_x + \psi_{xxx} = 0$ can be derived from Zermelo condition in Kawaguchi space of order 3. Moreover, a maximum order of differentiation in soliton equations corresponds to the order of Kawaguchi space. Thus, the soliton systems can geometrically connect with Kawaguchi space.

Finally, let us remark that the present approach can apply to a geometric theory of nonlinear dynamical systems called KCC-theory, because Zermelo condition can be regarded as a second variational equation in the KCC-theory. Therefore, the higher-order geometry is also useful in analysis of nonlinear dynamical systems.